Neutron interference: general theory of the influence of gravity, inertia and space-time torsion

1983 J. Phys. A: Math. Gen. 16 2457

(http://iopscience.iop.org/0305-4470/16/11/017)

J. Phys. A: Math. Gen. 16 (1983) 2457–2477. Printed in Great Britain

Neutron interference: general theory of the influence of gravity, inertia and space–time torsion

Jürgen Audretsch and Claus Lämmerzahl
Fakultät für Physik der Universität Konstanz, Postfach 5560, D-7750 Konstanz, West Germany

Received 30 September 1982, in final form 7 February 1983

Abstract. The general theory of the influence of gravity, inertia (i.e. interferometer motion) and space–time torsion on the outcome of neutron interference experiments is presented. The exact results are obtained in a general relativistic treatment based on the description of a stationary working interferometer in Riemann–Cartan space–time and on the WKB approximation for the neutron waves. Particular attention is paid to the influence on the spinor amplitude. There are two types of resulting amplitude effects; one originates in the non-integrability of the spinor connection and represents the influence of a modified Riemann–Cartan curvature; the other is caused by the influence of the interferometer rotation and acceleration and of space–time torsion during the time interval between the emission of the two coherent neutron waves. For practical purposes small effects are treated in an approximation. Two examples of a global evaluation of the expressions are given. Applications including a gravitational Aharonov–Bohm effect are discussed.

1. Introduction

During recent years increasing efforts have been made to incorporate quantum-mechanical concepts into general relativity. Motivated by the intriguing possibility of a unified theory of all physical interactions, quantum mechanics and quantum field theory in given curved space–times, quantised gravity, supersymmetric formulations, gravity and topology and similar subjects have been studied. Despite considerable efforts in the formulation and construction of quantum gravity and supergravity, the empirical basis of all these approaches is extremely limited and still on a level which is a long way from today's sophisticated theoretical research. Furthermore, the development of a theory of the empirical verifications of possible influences of gravity on quantum-mechanical systems is only just beginning. The aim of this paper is to contribute to such a theory. We want to explain the principles involved and outline the methods employed for the example of the influence of gravity, inertia and space–time torsion on the outcome of a neutron interference experiment.

Before doing so, let us briefly summarise some aspects of neutron interferometry. The field of interferometry in the Ångstrom region was opened for x-ray photons by Bonse and Hart (1965). The perfect-crystal neutron interferometer was first realised by Rauch *et al* (1974). As far as basic research is concerned, the important property of neutron interference is the fact that the coherent neutron beams represent a macroscopic quantum system† which can be influenced on a macroscopic scale. It is this property which has made it possible for the first time to detect the influence of the Earth's gravitational field on a quantum-mechanical system in the genuine quantum-mechanical effect of a phase shift (Colella *et al* 1975). Other important experiments in the context of basic research are the detection of the influence of the rotation of the Earth on the phase of neutrons (Werner *et al* 1979), the observation of the sign change of spinors under dynamically induced 2π rotations (Rauch *et al* 1975, Werner *et al* 1975; see also Rauch *et al* 1978) and the neutron analogue of the Fizeau experiment (Klein *et al* 1981). For a survey of the theoretical and experimental literature we refer to the proceedings of a workshop (Bonse and Rauch 1979) and to the reviews of Rauch and Petraschek (1978) and Greenberger and Overhauser (1979).

A neutron interferometer is sketched in figure 1. A single crystal of silicon has been cut to form three connected slabs. The neutron wave is split into two coherent parts by the splitter s, conducted over separate beam paths I and II and superimposed behind the mixer m. Phase shifts and spinor transformation of the wavefunctions may be induced, via the paths I and II, leading to intensity and polarisation effects in the resulting interference pattern. A space–time diagram of the interference experiment is given in figure 2. The tube represents the worldlines of the interferometer with tangent vectors u^α. S and M are the worldlines of splitter and mixer. The two coherent beams with four-velocity v^α interfere along the worldline M. In general, two beams interfering, for instance, in the space–time point B will have left the splitter at different worldpoints A_0 and A_1, separated by a time delay.

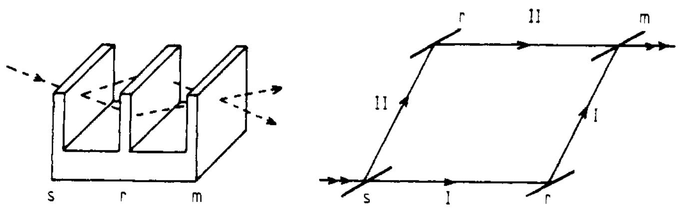

Figure 1. Neutron interferometer.

Let us recall that, as far as the influences of the Earth's gravity and rotation on the neutron phase are concerned, the outcome of the experiments above can be sufficiently described using the Schrödinger equation, the Euclidean theory of non-inertial reference frames and Newton's theory of gravity. This characterises today's status of empirical verification. In this paper we want to do the next step in elaborating a complete theory of the simultaneous influence of gravity, rotation, acceleration and space–time torsion on the phase as well as on the spinor amplitude (spinor polarisation) of the coherent neutron waves. This can be done within the WKB approximation in an exact way. For practical purposes the respective results may be approximated. An exact treatment of the influence on the spinor phase only has been given by Anandan (1977). The influence on the spinor amplitude has been discussed in the framework

† Macroscopic quantum systems, like superconductors and superfluid helium, seem to be the best candidates for the detection of gravity-induced quantum effects (Audretsch 1981a).

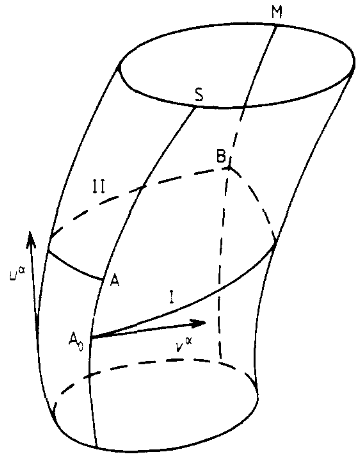

Figure 2. Space–time diagram of the worldlines of the interferometer and two coherent neutron beams. S is the four-worldline of the neutron source and the directly attached splitter. M is the four-worldline of the mixer. Neutron beams I and II interfering behind the mixer at the worldpoint B, have left the splitter at A_0 and A_1. The tangent vectors to the interferometer congruence and the neutron trajectories are u^α and v^α.

of non-relativistic Pauli theory by Eder and Zeilinger (1976) and by Zeilinger (1976). We extent the discussions by a general relativistic treatment of the spinor amplitude.

In a metric theory of gravitation, space–time geometry on one hand and gravitational and inertial forces on the other hand are unified. These two types of forces are thereby abolished. But, although they are no longer primary concepts, they can still be recognised in the relative acceleration of freely falling particles and in the fact that the kinematics of a field of observers influences their respective measurements in a specific manner. Accordingly, neutron interference may depend on the kinematical behaviour of the interferometer and on curvature, torsion and topology of the space–time.

The purpose of this paper is to discuss the related effects. We first present in § 2 an exact theory of the interference experiment in Riemann–Cartan space. It is based on the WKB approximation of the Dirac theory as given in § 2.1 and on the space–time description of stationary working interferometers in § 2.2. The resulting interfering spinorial wavefunctions are obtained in § 2.3. The expression for the phase shift is given in § 2.4. Via the amplitude, rotation and acceleration can influence interference patterns only in the case of non-vanishing time delay. This time delay is worked out in § 2.5. Section 3 contains the approximation for small time delays, small interferometer areas and non-relativistic motion. Two particular examples of a global evaluation of the exact expressions are treated in § 4. To demonstrate how the results may be used, we discuss some applications in § 5. The gravitational Aharonov–Bohm effect of § 5.3 shows that topology too may influence the interference. Finally, our conclusions are specified in § 6. To complete our treatment, we give in appendix 1 two theorems of the theory of time-like congruences which we need for the discussion of interferometer kinematics. How a stationary working neutron source is to be described can be read off directly from the definition of the spinor Lie derivative as given in appendix 2. The fact that interferometers, as described in § 2, really lead to a stationary interference pattern is shown in appendix 3.

2. Exact theory of the neutron interference experiment

2.1. *Propagation equations*

To describe the influence of space–time torsion, gravitational and inertial forces on neutrons, we use the Dirac theory in Riemann–Cartan space-time. The respective field equation is†

$$\mathrm{i}\hbar\gamma^{\mu}(\overset{\{\}}{\nabla}_{\mu}+\tfrac{1}{2}K_{[\mu\rho\sigma]}G^{\rho\sigma})\psi-m\psi=0 \qquad G^{\alpha\beta}=\tfrac{1}{2}\gamma^{[\alpha}\gamma^{\beta]}. \tag{2.1a, b}$$

Torsion $S_{\alpha\beta}{}^{\gamma}=\Gamma_{[\alpha\beta]}{}^{\gamma}$ enters the field equation (2.1) by the contortion tensor $K_{\alpha\beta}{}^{\gamma}=-S_{\alpha\beta}{}^{\gamma}+S_{\beta}{}^{\gamma}{}_{\alpha}-S^{\gamma}{}_{\alpha\beta}$. Metric and torsion act as external fields.

We assume that the neutrons travelling in the interferometer may be described in the WKB approximation. The respective propagation equations have been derived by Audretsch (1981b). By inserting the WKB ansatz

$$\psi(x)=\exp(\mathrm{i}\phi(x)/\hbar)\sum_{n=0}^{\infty}(-\mathrm{i}\hbar)^{n}a_{n}(x) \tag{2.2}$$

into the Dirac equation (2.1a) and equating the coefficients of the different orders of $\hbar$ to zero, we obtain the Hamilton–Jacobi equation $(\partial_{\alpha}\phi)(\partial^{\alpha}\phi)=m^{2}$ for the phase $\phi(x)$. The time-like congruence orthogonal to the hypersurfaces of constant phase $\phi(x)$ is given by the unit vector field $v_{\alpha}=-m^{-1}\partial_{\alpha}\phi$. It agrees in the classical limit with the normalised Dirac current j^{α} with $(j_{\varepsilon}j^{\varepsilon})^{-1/2}j^{\alpha}=v^{\alpha}+\mathrm{O}(\hbar)$ and describes the neutron 'rays'. The respective propagation is geodesic with regard to the Christoffel connection (extremal path): $v^{\alpha}{}_{;\varepsilon}v^{\varepsilon}=0$. The four-momentum p^{α} of the neutrons is introduced as

$$p_{\alpha}=mv_{\alpha}=-\partial_{\alpha}\phi \qquad p_{\alpha}p^{\alpha}=m^{2}. \tag{2.3a, b}$$

Note that for the WKB approximation $\psi=\exp(\mathrm{i}\phi(x)/\hbar)a_{0}(x)$ there is no influence of the torsion on the phase propagation. Torsion cannot be measured this way.

For the propagation of the WKB spinor amplitude a_0 along the v^{α} lines we then obtain after decomposing a_0 with regard to the normalised spinor b_0 according to

$$a_{0}=f(x)b_{0}(x) \qquad \overline{b_{0}}b_{0}=1 \tag{2.4a, b}$$

the two *propagation equations*

$$(\partial_{\alpha}f)v^{\alpha}=-\tfrac{1}{2}\theta_{v}f \qquad \theta_{v}=v^{\alpha}{}_{;\alpha} \tag{2.5a, b}$$

and

$$(\partial_{\alpha}b_{0}+\tilde{\Gamma}_{\alpha}b_{0})v^{\alpha}=0 \tag{2.6}$$

† *Notations and conventions.* We use $c=1$, unless c is explicitly inserted into an equation. Signature of metric tensor $g_{\alpha\beta}$: $(+,-,-,-)$. $\alpha,\beta,\ldots=0,\ldots,3$ are tensor indices raised and lowered with $g_{\alpha\beta}$. $a,b,\ldots=0,\ldots,3$ and $\hat{a},\hat{b}\ldots=1,2,3$ are tetrad indices raised and lowered with $\eta_{ab}=\mathrm{diag}(+1,-1,-1,-1)$. Particular values of $a,b,\ldots$ are denoted by brackets: $A^{(1)}=A^{a=1}$. Symmetrisation: $A_{(\alpha\beta)}=\frac{1}{2}(A_{\alpha\beta}+A_{\beta\alpha})$. Antisymmetrisation: $A_{[\alpha\beta]}=\frac{1}{2}(A_{\alpha\beta}-A_{\beta\alpha})$ and $A_{[\alpha\beta\gamma]}=\frac{1}{3}(A_{\alpha[\beta\gamma]}+A_{\beta[\gamma\alpha]}+A_{\gamma[\alpha\beta]})$. Projection is denoted by a bold face index: $A^{\boldsymbol{\alpha}}=h^{\alpha}_{\beta}A^{\beta}$ with $h^{\alpha}_{\beta}=\delta^{\alpha}_{\beta}-u^{\alpha}u_{\beta}$. The covariant tensor derivative with regard to the Christoffel connection is denoted by a semicolon: ( ). The corresponding spinor derivative is $\overset{\{\}}{\nabla}_{\mu}=\partial_{\mu}+\overset{\{\}}{\Gamma}_{\mu}$ with $\overset{\{\}}{\Gamma}_{\mu}=-\frac{1}{2}h^{\varepsilon}_{a;\mu}h_{b\varepsilon}G^{ab}$ and $G^{ab}=\frac{1}{2}\gamma^{[a}\gamma^{b]}$ where h^{α}_{a} is a tetrad $(h^{\alpha}_{a}h^{\beta}_{b}g_{\alpha\beta}=\eta_{ab})$ and γ^{a} of $\gamma^{\alpha}=h^{\alpha}_{a}\gamma^{a}$ are the standard Dirac matrices: $\gamma^{(a}\gamma^{b)}=\eta^{ab}$.

where we have introduced the *new connection*

$$\tilde{\Gamma}_{\mu\gamma}{}^{\lambda} = \left\{ {\lambda \atop \mu\gamma} \right\} - 3K_{[\mu\gamma\varepsilon]}g^{\varepsilon\lambda} \qquad \tilde{\Gamma}_{\alpha} = \overset{\{\}}{\Gamma}_{\alpha} + \tfrac{3}{2}K_{[\alpha\kappa\lambda]}G^{\kappa\lambda}. \tag{2.7a, b}$$

It contains only the totally antisymmetric part of the torsion. θ_v represents the expansion of the neutron rays and equation (2.5) relates this expansion to the change of the intensity of the neutron ray. Equation (2.6) shows that the normalised spinor amplitude b_0 is parallely propagated with regard to the new spinor connection $\tilde{\Gamma}_{\alpha}$ which, in Riemann–Cartan space, governs the WKB limit of the dynamical spinor propagation. The components of the torsion which enter our calculation can be combined into the axial vector part of the torsion $K^{\mu} = \frac{1}{6}\eta^{\mu\alpha\beta\gamma}K_{[\alpha\beta\gamma]}$ with $\eta^{\alpha\beta\gamma\delta} = h^{\alpha}_{a}h^{\beta}_{b}h^{\gamma}_{c}h^{\delta}_{d}\varepsilon^{abcd}$ and $\varepsilon^{(0)(1)(2)(3)} = 1$.

The physics of the neutron field is described by the Dirac current

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi = f^2 v^{\mu} + O(\hbar) \tag{2.8}$$

and by the spin vector

$$S^{\mu} = \bar{\psi}\gamma^{\mu}\gamma^{5}\psi = f^2 S^{\mu} + O(\hbar) \qquad \hat{S}^{\mu} = \overline{b_0}\gamma^{\mu}\gamma^{5}b_0. \tag{2.9a, b}$$

$\gamma^5 = \mathrm{i}\gamma^{(0)}\gamma^{(1)}\gamma^{(2)}\gamma^{(3)}$. The normalised spin vector $\hat{S}^{\mu}$ represents the direction of the neutron polarisation in the rest-space of the particle ($\hat{S}_{\mu}v^{\mu} = 0$).

2.2. *Interferometer and neutron source*

In this section we will discuss the requirements which the external fields, the interferometer and the neutron source have to fulfil, as well as the instructions for the performance of the experiment. The basic intention is thereby to introduce an experimental arrangement in a Riemann–Cartan space–time which guarantees a stationary interference pattern.

2.2.1. Stationary interferometer. The 4-worldlines of the matter elements (compare with figure 2) which constitute the interferometer body form a time-like congruence with normalised tangent vectors $u^{\alpha}(x)$. Because macroscopic bodies with no net elementary-particle spin are not influenced by torsion (see, for example, Yasskin and Stoeger 1980), the kinematical properties of this congruence can be described with reference to the Christoffel connection only. We demand a stationary interferometer which is characterised by the following three requirements (compare with appendix 1): (i) rigidity of the interferometer $\leftrightarrow$(A1.1i), (ii) constant interferometer rotation$\leftrightarrow$(Al.2ii) and (iii) co-rotation of all non-gravitational forces on interferometer elements causing the deviation from free fall$\leftrightarrow$(A1.2iii).

According to the two lemmas of appendix 1, the consequence of the physical requirements (i)–(iii) is that the interferometer congruence is isometric

$$u^{\alpha} = \mathrm{e}^{-U}\xi^{\alpha} \qquad \underset{\xi}{\mathscr{L}}g_{\alpha\beta} = 2\xi_{(\alpha;\beta)} = 0 \qquad \underset{\xi}{\mathscr{L}}u^{\alpha} = 0 \tag{2.10a, b, c}$$

and that the acceleration a^{α} has a potential $U(x)$ (compare with A1.4))

$$a_{\alpha} = -\partial_{\alpha}U(x) \qquad (\partial_{\alpha}U(x))u^{\alpha} = 0. \tag{2.11a, b}$$

For a non-freely falling interferometer, the potential $U(x)$ consists of the gravitational and centrifugal potentials.

Because of (2.11*a*) the potential $U(x)$ is constant along the interferometer's worldlines and may, therefore, be gauged according to

$$U \overset{S}{=} 0 \tag{2.12}$$

on the worldline S of the neutron source.

2.2.2. Stationary neutron source. We represent the source which produces the polarised neutron beam by a co-moving tetrad $\check{h}_a^\alpha$ along the worldline S with $\check{h}_{(0)}^\alpha \overset{S}{=} u^\alpha$ and $\check{h}_a^{\hat{a}}(\mathrm{S})$ ($\hat{a} = 1, 2, 3$) pointing always to the same neighbouring element of the interferometer so that $\check{h}_a^\alpha$ rotates and accelerates in the same way as the interferometer. The rigidity of the interferometer congruence makes such a choice possible.

The u^α congruence being isometric (2.10), we have $\mathscr{L}_\xi \check{h}_{(0)}^\alpha \overset{S}{=} 0$. Orthogonal connecting vectors of an isometric congruence are Lie transferred which implies $\mathscr{L}_\xi \check{h}_{\hat{a}}^\alpha \overset{S}{=} 0$, so that the complete co-moving tetrad field $\check{h}_a^\alpha(S)$ is obtained by Lie propagation.

That the neutron source is working stationary means that the spinor amplitude $a_0(\mathrm{S})$ of § 2.1 does not change numerically when referred to the co-moving tetrad $\check{h}_a^\alpha(\mathrm{S})$ and is, therefore, obtained by co-dragging with the Lie-displaced tetrad $\check{h}_a^\alpha$ (Lie transfer of spinor, compare with appendix 2). This implies that the spinor Lie derivative of a_0 along S with regard to the Killing vector ξ vanishes: $\mathscr{L}_\xi a_0 \overset{S}{=} 0$.

With the decomposition (2.4) and because of (A2.8*b*) we find

$$\underset{\xi}{\mathscr{L}} f \overset{S}{=} 0 \tag{2.13}$$

and therefore

$$\underset{\xi}{\mathscr{L}} b_0 \overset{S}{=} 0 \qquad (\partial_\alpha b_0)u^\alpha \overset{S}{=} [-\overset{\{\}}{\Gamma}_\alpha u^\alpha - \tfrac{1}{2}(\partial_{[\alpha}\xi_{\beta]})G^{\alpha\beta}]b_0, \tag{2.14a, b}$$

whence, along S the absolute value f of the generated spinor is constant and its spinor direction b_0 is Lie transferred. The second equation is obtained with (A2.7). It represents the propagation equation for b_0 along the u^α worldline S. This completes the propagation equations (2.5) and (2.6). We note that the neutron four-velocity is also Lie propagated along S (compare with (A3.3)). It follows from (2.3) and (2.10) that the neutrons must be generated with constant energy $E = p_\alpha u^\alpha$:

$$(\partial_\alpha E)u^\alpha \overset{S}{=} 0. \tag{2.15}$$

2.2.3. Beam splitting and bending. The polarised incident neutron beam is split at the splitter s (compare with figure 1) in two coherent polarised beams denoted by I and II. For reasons of simplicity they are assumed to be of equal intensity:

$$f_{\mathrm{I}} = f_{\mathrm{II}}. \tag{2.16}$$

Furthermore, it is assumed that the two beams do not spread, that is $\theta_v = 0$, so that according to (2.5) the respective f remains constant along the two paths:

$$(\partial_\alpha f_{\mathrm{I,II}})v^\alpha = 0. \tag{2.17}$$

Between splitter, mirrors and mixer the neutron waves are moving freely. The respective spinor is parallely propagated according to (2.6). In addition, there is a change of the propagation direction of the two beams at splitter, mirrors and mixer which is a four-rotation causing a spin transformation. We assume that the effects of all these spin transformations cancel (i.e. add up to zero) and that the mirrors act point-wise without changing energy, so that the neutron paths may be represented by piece-wise geodesic worldlines ((2.25*b*) is valid except for single points). So we may represent the four worldlines of the two beams as in figure 2.

If, in addition to all the conditions stated above, the space–time torsion is stationary as well, then the resulting interference pattern in M does not change in time. See appendix 3 for proof.

2.3. *Interfering wavefunctions*

The differential equations (2.6) and (2.14*b*) for the two spinor propagations and their solution in question are both of the type

$$\mathrm{d}b_0/\mathrm{d}s = \mathscr{A}(s)b_0(s) \qquad b_0(s) = \mathscr{T}\left(\exp\int_{s_0}^{s}\mathscr{A}(s')\,\mathrm{d}s'\right)b_0(s_0) \tag{2.18a, b}$$

where the one-parameter family $\mathscr{A}(s)$ of operators is proportional to the generators $G^{\alpha\beta}$ of spin transformations. The monotonically increasing parameter s is the arc length of the respective time-like worldline. $\mathscr{T}$ is the usual time-ordering operator with regard to s.

The resulting wavefunction $\psi(\mathrm{B})$ in B is obtained by the interference of the two WKB solutions ψ_{I} and ψ_{II} reaching B along path I and path II: $\psi(\mathrm{B}) = \psi_{\mathrm{I}}(\mathrm{B}) + \psi_{\mathrm{II}}(\mathrm{B})$.

To work out $\psi(\mathrm{B})$ we will relate the spinor $\psi_{\mathrm{II}}(\mathrm{B})$ along the paths $\mathrm{BA_1}$, $\mathrm{A_1A_0}$, $\mathrm{A_0B}$ with the spinor $\psi_{\mathrm{I}}(\mathrm{B})$ using the respective propagation equations. With (2.3) for the phase, (2.13), (2.16) and (2.17) for f, (2.6), (2.14*b*) and (2.18) for b_0 and the WKB structure $\psi = \exp(\mathrm{i}\phi/\hbar)fb_0$ we obtain the basic relation

$$\psi_{\mathrm{II}}(\mathrm{B}) = \exp(\mathrm{i}\Delta\phi/\hbar)\mathscr{T}\exp\left\{-\oint\tilde{\Gamma}_\alpha\,\mathrm{d}x^\alpha + \int_{\mathrm{A_0}}^{\mathrm{A_1}}[\tfrac{1}{2}(\omega_{\sigma\tau} + 2a_{[\sigma}u_{\tau]}) + \tfrac{3}{2}K_{[\alpha\sigma\tau]}u^\alpha]G^{\sigma\tau}\,\mathrm{d}s\right\}\psi_{\mathrm{I}}(\mathrm{B}) \tag{2.19}$$

where we have inserted (A1.5) along $\mathrm{A_0A_1}$. The ring integral is defined as $\oint = +\int_{\mathrm{A_0}}^{\mathrm{A_1}} + \int_{\mathrm{A_1}}^{\mathrm{B}} + \int_{\mathrm{B}}^{\mathrm{A_0}}$. The resulting phase difference $\Delta\phi$ is obtained by integration according to

$$\Delta\phi = \oint \partial_\alpha\phi\,\mathrm{d}x^\alpha = -\oint p_\alpha(x)\,\mathrm{d}x^\alpha. \tag{2.20}$$

It is not demanded that space–time be simply connected. The interference loop may, for instance, encircle a singularity.

The equation (2.19) reflects the following structure. There is no influence of the torsion on the phase $\Delta\phi$. The first integral in (2.19) represents the non-integrability of the spinor amplitude which is caused by the non-vanishing of the generalised curvature related to the spinor connection $\tilde{\Gamma}_\alpha$. The existence of the second integral in (2.19) goes back to the fact that, in general, the points $\mathrm{A_0}$ and $\mathrm{A_1}$ do not agree. There will be a time delay between the emission of ψ_{I} and ψ_{II}. During this time interval the source may rotate with $\omega_{\sigma\tau}$ and accelerate with a_σ. Because the spinor is co-moving with the source, rotation and acceleration cause spin transformations which are reflected by the first two terms in the integral $\int_{A_0}^{A_1}$. An influence of the interferometer motion on the spinor amplitude (which could be called an inertial effect in the Euclidean notation) can be registered if and only if there is a time delay. The fact that the contortion $K_{\alpha\sigma\tau}$ appears in the integral $\int_{A_0}^{A_1}$ has primarily a technical reason; because the neutron source is not influenced by torsion, the co-dragging of the spinors from A_0 to A_1 refers to the Christoffel connection only. To obtain the ring integral $\oint$ in (2.19), one has to introduce a compensating term in $\int_{A_0}^{A_1}$. The resulting interpretation is, therefore, that torsion affects our result in a twofold way, firstly as part of the curvature of the tilded connection (2.7) and secondly in a linear way as a totally antisymmetric contortion along the path A_0A_1 if there is a time delay. In the latter case it acts like a rotation of the neutron source, as can be seen in (2.19). There is no acceleration-like contribution to the torsion.

2.4. *Phase shift*

The phase shift $\Delta\phi$ of (2.20) has already been discussed by Anandan (1977) in analogy to the treatment of the electromagnetic Sagnac effect by Ashtekar and Magnon (1975). We briefly sketch the proof of the exact relation for $\Delta\phi$ which permits the computation of $\Delta\phi$.

We introduce the three-propagation of the neutron beam in the rest space of the interferometer

$$n^\mu = (-v^\alpha v_\alpha)^{-1/2} v^\mu \qquad n^\alpha n_\alpha = -1 \qquad n_\alpha u^\alpha = 0. \qquad (2.21a, b, c)$$

Decomposition of the neutron four-momentum p^α leads to

$$p^\alpha = E u^\alpha + p n^\alpha \qquad E^2 - p^2 = m^2 \qquad (2.22a, b)$$

where $E = m(1-v^2)^{-1/2}$ is the total energy, $p = mv(1-v^2)^{-1/2}$ the three-momentum, and v the absolute value of the neutron three-velocity, all as measured in the interferometer's rest space.

The energy E has the constant value $E(\mathrm{A})$ on the worldline S of the source because of (2.15). On the other hand, the p^α streamlines are geodesic and $\xi^\alpha(x)$ of (2.10) is a Killing vector field. Accordingly, $p_\alpha \xi^\alpha$ is constant along the neutron streamlines. Taking these two facts together, we have the result that $p_\alpha \xi^\alpha$ is constant on the whole two-dimensional tube of the interferometer worldlines in the space–time diagram of figure 2 (the neutron worldlines lie on this tube).

$$p_\alpha \xi^\alpha = p_\alpha u^\alpha \, \mathrm{e}^{U(x)} = E(x)\, \mathrm{e}^{U(x)} = E(\mathrm{A}) \qquad (2.23)$$

where we have used the normalisation (2.12). Accordingly we find with (2.22) for $\Delta\phi$ the resulting phase shift of (2.20)

$$\Delta\phi = -E(\mathrm{A}) \oint \mathrm{e}^{-U(x)} u_\alpha \, \mathrm{d}x^\alpha - \left(\int_{\mathrm{II}} - \int_{\mathrm{I}} \right) p(x) n_\alpha \, \mathrm{d}x^\alpha \qquad (2.24)$$

where $\int_{\mathrm{I}}$ and $\int_{\mathrm{II}}$ denote the integrals $\int_{A_0}^{B}$ and $\int_{A_1}^{B}$.

If the space–time region is simply connected and the interferometer worldlines fill the interior of the tube, as is usually the case in all realistic laboratory situations, the Killing field ξ^α extends to this region and we may use Stoke's theorem to obtain the phase difference:

$$\Delta\phi = E(\mathrm{A}) \int_\Sigma e^{-U(x)}\omega_{\alpha\beta}\, d\sigma^{\alpha\beta} - \left(\int_{\mathrm{II}} - \int_{\mathrm{I}}\right) p(x) n_\alpha\, dx^\alpha$$
$$p(x) = (E(\mathrm{A})^2 e^{-2U(x)} - m^2)^{1/2} \qquad (2.25a, b)$$

with the momentum $p(x)$ of the neutrons as measured in the interferometer rest space.

2.5. *Time delay*

In general there will be a time delay

$$\Delta s = \int_{\mathrm{A}_0}^{\mathrm{A}_1} ds = \int_{\mathrm{A}_0}^{\mathrm{A}_1} u_\alpha\, dx^\alpha \qquad (2.26)$$

between the emission of the two coherent neutron waves at A_0 and A_1. We can relate this time delay Δs to the phase shift $\Delta\phi$ by

$$\Delta\phi = -\oint p_\alpha\, dx^\alpha = -E(\mathrm{A})\,\Delta s - m\left(\int_{\mathrm{II}} - \int_{\mathrm{I}}\right) v_\alpha\, dx^\alpha. \qquad (2.27)$$

Using (2.23) in the form $m\, e^{U(x)}/E(\mathrm{A}) = (u_\alpha v^\alpha)^{-1}$ and noting that $(u_\varepsilon v^\varepsilon) v_\alpha\, dx^\alpha = u_\alpha\, dx^\alpha$ is valid along the neutron paths $\mathrm{A}_1\mathrm{B}$ and BA_0, we get

$$\Delta\phi = -E(\mathrm{A})\,\Delta s - \frac{m^2}{E(\mathrm{A})}\left(\int_{\mathrm{II}} - \int_{\mathrm{I}}\right) e^{U(x)} u_\alpha\, dx^\alpha. \qquad (2.28)$$

We complete the last term to a ring integral and solve the resulting equation for Δs to find the intended relation between time delay Δs and phase $\Delta\phi$:

$$\Delta s = -\frac{E(\mathrm{A})}{p(\mathrm{A})^2}\Delta\phi - \frac{m^2}{p(\mathrm{A})^2}\oint e^{U(x)} u_\alpha\, dx^\alpha \qquad (2.29)$$

which renders with (2.24) the following exact equation for the time delay

$$\Delta s = \oint e^{-U(x)} u_\alpha\, dx^\alpha - \frac{2m^2}{p(\mathrm{A})^2}\oint (\sinh U(x)) u_\alpha\, dx^\alpha + \frac{E(\mathrm{A})}{p(\mathrm{A})^2}\left(\int_{\mathrm{II}} - \int_{\mathrm{I}}\right) p(x) n_\alpha\, dx^\alpha \qquad (2.30)$$

with $p(x)$ of (2.25). It is evident from (2.30) that the time delay has nothing to do with the closing gap of an infinitesimal parallelogram in a space–time with torsion. There is no such influence of torsion on the resulting interference effect. Again, if the Killing field extends to the whole interior of the two-dimensional tube on which the neutron worldlines are lying, we may relate the first integral to the rotation $\omega_{\alpha\beta}$ of the interferometer.

3. Interference: the infinitesimal case

3.1. *Interfering wavefunctions*

The appearance of the time-ordering operator $\mathcal{T}$ in (2.19) makes a further simplification of this equation difficult. But there are two particular physical situations in which $\mathcal{T}$ may be omitted: one is the *global case*, where the operators $\mathcal{A}(s)$ of (2.18) commute for all values of s. This will be dealt with in § 4. The other situation, which is characterised by small resulting interference effects, will be called the *infinitesimal case*. It can be found in a simply connected space–time region for small time delays Δs and small area $\Delta\Sigma^{\alpha\beta}$

$$\Delta\Sigma^{\alpha\beta} = \int_\Sigma d\sigma^{\alpha\beta} \qquad \Sigma_\alpha = \tfrac{1}{2}\eta_{\beta\alpha\gamma\delta}u^\beta\,\Delta\Sigma^{\gamma\delta} \qquad \Sigma = (-\Sigma_\alpha\Sigma^\alpha)^{1/2} \tag{3.1a, b, c}$$

of the interferometer. In the infinitesimal case the following integrals may be approximated according to

$$\int_\Sigma e^{-U}\omega_{\alpha\beta}\,d\sigma^{\alpha\beta} = \omega_{\alpha\beta}\,\Delta\Sigma^{\alpha\beta} = -2\omega_\alpha\Sigma^\alpha \tag{3.2}$$

$$\int_{A_0}^{A_1}[\tfrac{1}{2}(\omega_{\sigma\tau}+2a_{[\sigma}u_{\tau]})+\tfrac{3}{2}K_{[\alpha\sigma\tau]}u^\alpha]G^{\sigma\tau}\,ds = [\tfrac{1}{2}(\omega_{\sigma\tau}+2a_{[\sigma}u_{\tau]})+\tfrac{3}{2}K_{[\alpha\sigma\tau]}u^\alpha]G^{\sigma\tau}\Delta s \tag{3.3}$$

$$\oint\tilde{\Gamma}_\alpha\,dx^\alpha = \tilde{R}_{\alpha\beta\kappa\lambda}G^{\kappa\lambda}\Delta\Sigma^{\alpha\beta} \tag{3.4}$$

where the modified spin curvature related to the tilded spinor connection is given by

$$\tilde{R}_{\alpha\beta\kappa\lambda}G^{\kappa\lambda} = 2\,\partial_{[\alpha}\tilde{\Gamma}_{\beta]}+2\tilde{\Gamma}_{[\alpha}\tilde{\Gamma}_{\beta]}. \tag{3.5}$$

The right-hand side of (3.2)–(3.4) is to be taken on the worldline S. Terms of order $(U/c^2)^2$ are neglected throughout this section. So the basic relation (2.19) reduces with (3.3) and (3.4) to

$$\psi_{\rm II}(B) = \exp(i\Delta\phi/\hbar)(1+\tfrac{1}{2}z_{\kappa\lambda}G^{\kappa\lambda})\psi_{\rm I}(B) \tag{3.6}$$

with

$$\tfrac{1}{2}z_{\kappa\lambda} = -\tilde{R}_{\kappa\lambda\mu\nu}\Delta\Sigma^{\mu\nu}+[\tfrac{1}{2}(\omega_{\kappa\lambda}+2a_{[\kappa}u_{\lambda]})+\tfrac{3}{2}K_{[\epsilon\kappa\lambda]}u^\epsilon]\Delta s. \tag{3.7}$$

According to (3.6) $\psi_{\rm II}(B)$ is obtained from $\psi_{\rm I}(B)$ by a change of the phase and an infinitesimal spin transformation related to the infinitesimal Lorentz transformation

$$L_\alpha{}^\beta = \delta_\alpha^\beta + z_\alpha{}^\beta \qquad z_{\alpha\beta} = z_{[\alpha\beta]}. \tag{3.8a, b}$$

We base our interpretation on this concept.

3.2. Phase shift and time delay

We reintroduce the velocity of light c and neglect the relativistic terms $O(v^2/c^2)$. Equation (2.25) for the phase shift reduces then with the neutron momentum of (2.25b)

$$p(x) = p(A)[1-(m^2U(x)/p(A))] \tag{3.9}$$

and with (3.2) to

$$\Delta\phi = -2m\omega_\alpha\Sigma^\alpha + p(A)\,\Delta l + m^2/p(A)\left(\int_{\rm II}-\int_{\rm I}\right)U(x)n_\alpha\,dx^\alpha \tag{3.10}$$

where we have introduced the difference Δl of the three-length of the two paths I and II $\Delta l = -(\int_{\rm II}-\int_{\rm I})n_\alpha\,dx^\alpha$ as measured in the rest space of the interferometer.

In the infinitesimal case, the time delay Δs of (2.30) reduces with (3.2) and (3.9) to

$$\Delta s = \frac{2}{c^2}\omega_\alpha \Sigma^\alpha - \frac{2}{v(\mathrm{A})^2 c}\oint U(x) u_\alpha \, \mathrm{d}x^\alpha - \frac{1}{v(\mathrm{A})}\left[\Delta l + \frac{1}{v(\mathrm{A})^2}\left(\int_{\mathrm{II}} - \int_{\mathrm{I}}\right) U(x) n_\alpha \, \mathrm{d}x^\alpha\right]. \tag{3.11}$$

Because of (2.12) the path $\mathrm{A}_0\mathrm{A}_1$ does not contribute to the ring integral. Along the remaining paths we have with (2.22) $u_\alpha \, \mathrm{d}x^\alpha = -[c/v(x)] n_\alpha \, \mathrm{d}x^\alpha$ so that we obtain for the time delay

$$\Delta s = \frac{2}{c^2}\omega_\alpha \Sigma^\alpha - \frac{\Delta l}{v(\mathrm{A})} + \frac{1}{v(\mathrm{A})^3}\left(\int_{\mathrm{II}} - \int_{\mathrm{I}}\right) U(x)\left(\frac{1 - \Delta v(x)/v(\mathrm{A})}{1 + \Delta v(x)/v(\mathrm{A})}\right) n_\alpha \, \mathrm{d}x^\alpha \tag{3.12}$$

where $\Delta v(x) = v(x) - v(\mathrm{A})$ denotes the change of velocity which the neutron experiences within the interferometer. The integrations refer to the interferometer rest space. Equation (3.12) shows that the time delay between the emission at A_0 and A_1 may be caused by the rotation of the interferometer by a differing length of the two paths in the interferometer and by different influences of the potential $U(x)$ on the neutron waves travelling along these two paths. Note that the influence of the rotation is a relativistic effect.

3.3. *Intensity and polarisation*

To obtain the intensity of the interfering waves $\psi_{\mathrm{I}}(\mathrm{B})$ and $\psi_{\mathrm{II}}(\mathrm{B})$ and the resulting polarisation behind the mixer at B, we have to insert $\psi(\mathrm{B}) = \psi_{\mathrm{I}}(\mathrm{B}) + \psi_{\mathrm{II}}(\mathrm{B})$ given by (3.6) into the Dirac current j^μ of (2.8) and the polarisation vector S^μ of (2.9). Using well known relations for products of Dirac matrices, this leads, after some calculations, to

$$j^\alpha = 2[(1 + \cos(\Delta\phi/\hbar)) j_{\mathrm{I}}^\alpha(\mathrm{B}) + (1 + \cos(\Delta\phi/\hbar))(\tfrac{1}{2} z^\alpha{}_\beta) j_{\mathrm{I}}^\beta(\mathrm{B}) - (\sin(\Delta\phi/\hbar))(\tfrac{1}{2} z^{*\alpha}{}_\beta) S_{\mathrm{I}}^\beta(\mathrm{B})] \tag{3.13}$$

$$S^\alpha = 2[(1 + \cos(\Delta\phi/\hbar)) S_{\mathrm{I}}^\alpha(\mathrm{B}) + (1 + \cos(\Delta\phi/\hbar))(\tfrac{1}{2} z^\alpha{}_\beta) S_{\mathrm{I}}^\beta(\mathrm{B}) - (\sin(\Delta\phi/\hbar))(\tfrac{1}{2} z^{*\alpha}{}_\beta) j_{\mathrm{I}}^\beta(\mathrm{B})] \tag{3.14}$$

where we have omitted terms $O(z_{\alpha\beta}{}^2)$ according to our infinitesimal approach and introduced the dual $z^*_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta}{}^{\kappa\lambda} z_{\kappa\lambda}$. $j_{\mathrm{I}}^\alpha(\mathrm{B})$ and $S_{\mathrm{I}}^\alpha(\mathrm{B})$ are four-current and four-polarisation of the neutron wave $\psi_{\mathrm{I}}(\mathrm{B})$ only, as it is leaving the mixer in B.

Apart from the phase shift $\Delta\phi$, the two quantities $j^\alpha(\mathrm{B})$ and $S^\alpha(\mathrm{B})$ depend on the matrix $z_{\mu\nu}$ of an infinitesimal Lorentz transformation explicitly given in (3.7). With regard to a normalised time-like vector u^α, which in our case may be the four-velocity of the interferometer†, the antisymmetric $z_{\mu\nu}$ can be decomposed into space-like vectors z_μ and ζ_μ:

$$z_{\mu\nu} = \boldsymbol{z}_{\mu\nu} + 2\zeta_{[\mu}u_{\nu]} \qquad z_\mu = \tfrac{1}{2}\eta_{\varepsilon\mu\kappa\lambda} z^{\kappa\lambda} u^\varepsilon = z^*_{\varepsilon\mu} u^\varepsilon. \tag{3.15a, b}$$

In the infinitesimal case the space-like vectors ζ_μ with $\zeta_\mu = z_{\mu\nu}u^\nu$ and z_μ are, because of (3.7), given by

$$\zeta_\mu = -2\tilde{R}_{\mu\varepsilon\kappa\lambda}\Delta\Sigma^{\kappa\lambda}u^\varepsilon + a_\mu \Delta s \tag{3.16}$$

† The calculation here and in § 4 can easily be generalised to a time-like vector which does not agree with u^α.

$$z_\mu = -2{}^*\tilde{R}_{\varepsilon\mu\kappa\lambda}\,\Delta\Sigma^{\kappa\lambda}u^\varepsilon + \tfrac{1}{2}(\omega_\mu - 3K_\mu)\,\Delta s \qquad {}^*\tilde{R}^{\mu\nu\kappa\lambda} = \tfrac{1}{2}\eta^{\mu\nu\rho\sigma}\tilde{R}_{\rho\sigma}{}^{\kappa\lambda}. \tag{3.17a, b}$$

These formulae show that the tilded curvature and the interferometer acceleration a_μ result in an infinitesimal Lorentz boost whereas the left dual of the tilded curvature together with the interferometer rotation ω_μ and three-axial vector K_μ of the torsion results in an infinitesimal Lorentz three-rotation.

Introducing for $\hat{S}^\mu$, ζ^μ, z^μ the corresponding Euclidean three-vectors $\hat{\boldsymbol{S}}$, $\boldsymbol{\zeta}$, $\boldsymbol{z}$ according to

$$\zeta_\mu v_{\rm I}^\mu (u_\varepsilon v_{\rm I}^\varepsilon)^{-1} = -\boldsymbol{\zeta}\cdot\boldsymbol{v}_{\rm I} \qquad z_\mu \hat{S}_{\rm I}^\mu = -\boldsymbol{z}\cdot\hat{\boldsymbol{S}}_{\rm I} \tag{3.18a, b}$$

one can calculate from (3.13) and (3.14) the measured intensity

$$J({\rm B}) = j^\mu({\rm B})u_\mu({\rm B}) \tag{3.19}$$

and polarisation $\hat{\boldsymbol{S}}$ using $J_{\rm I}({\rm B}) = j_{\rm I}^\mu({\rm B})u_\mu = f_{\rm I}^2$ and neglecting effects of $O(v^2/c^2)$:

$$\begin{aligned} J({\rm B})/4J_{\rm I}({\rm B}) = \tfrac{1}{2}[&1+\cos(\Delta\phi/\hbar) + (1+\cos(\Delta\phi/\hbar))(2c)^{-1}\boldsymbol{\zeta}\cdot\boldsymbol{v}_{\rm I}({\rm B}) \\ &+ (\sin(\Delta\phi/\hbar))\tfrac{1}{2}\boldsymbol{z}\cdot\hat{\boldsymbol{S}}_{\rm I}({\rm B})] \end{aligned} \tag{3.20}$$

$$\begin{aligned} \hat{\boldsymbol{S}}({\rm B}) = (4J_{\rm I}({\rm B})/J({\rm B}))\{&\tfrac{1}{2}(1+\cos(\Delta\phi/\hbar))[\hat{\boldsymbol{S}}_{\rm I}({\rm B}) - \tfrac{1}{2}\hat{\boldsymbol{S}}_{\rm I}(B)\times\boldsymbol{z} + (2c)^{-1}\boldsymbol{\zeta}(\hat{\boldsymbol{S}}_{\rm I}(B)\cdot\boldsymbol{v}_{\rm I}(B))] \\ &+ \tfrac{1}{2}(\sin(\Delta\phi/\hbar))[\tfrac{1}{2}\boldsymbol{z} - (2c)^{-1}(\boldsymbol{\zeta}\times\boldsymbol{v}_{\rm I}({\rm B}))]\}. \end{aligned} \tag{3.21}$$

The reference quantities with the index I are obtained experimentally by measuring intensity $J_{\rm I}$, polarisation $S_{\rm I}$ and velocity $v_{\rm I}$ of the beam behind the mixer after screening beam II. The influence on the spinor amplitude is reflected by the rotation $\boldsymbol{z}$, and the boost $\boldsymbol{\zeta}$. Measurements of intensity and polarisation represent two independent tests of the influence of gravity, kinematics and space–time torsion on phase and spinor amplitude.

4. Interference: the global case

We assume that the operator $\mathcal{A}(s)$ of (2.18) commutes for all points on the path $A_0BA_1A_0$:

$$\mathcal{A}(s_1)\mathcal{A}(s_2) = \mathcal{A}(s_2)\mathcal{A}(s_1). \tag{4.1}$$

In this case the integration can be accomplished without the time-ordering operator $\mathcal{T}$. There are no restrictions as far as the connectedness of the space–time is concerned. The operator $\mathcal{A}(s)$ in the propagation equations (2.6) and (2.14*b*) is a linear combination of components of $G^{\mu\nu}$. Therefore, assuming (4.1), the integration (2.19) results in a solution of the structure

$$\psi_{\rm II} = \exp({\rm i}\Delta\phi/\hbar)\exp(\tfrac{1}{2}z_{\mu\nu}G^{\mu\nu})\psi_{\rm I}({\rm B}) \tag{4.2}$$

with $z_{\mu\nu}$ being no longer an infinitesimal quantity, but the result of a global integration. Accordingly, we cannot expect a generally applicable formula like (3.7) which relates $z_{\mu\nu}$ to the external fields. Any physical situation needs a new explicit integration. Nevertheless, we can work out the resulting interference for two classes of physical situations: the resulting $z_{\mu\nu}$ in (4.2) is either a pure Lorentz three-rotation or a pure Lorentz boost. We use the decomposition of $z_{\mu\nu}$ and the related definitions of § 3.3.

(*a*) *Pure Lorentz three-rotation.* This case is characterised by $z_{\mu\nu} = z_{\mu\nu}$ and we can evaluate (4.2) using

$$(z_{\alpha\beta}G^{\alpha\beta})^{2m} = (-1)^m z^{2m} \qquad (z_{\alpha\beta}G^{\alpha\beta})^{2m+1} = (-1)^m z^{2m+1}\hat{z}_{\alpha\beta}G^{\alpha\beta} \tag{4.3a, b}$$

($m = 0, 1, 2, \ldots$) where $\hat{z}_{\alpha\beta} = z^{-1}z_{\alpha\beta}$, $z = (\frac{1}{2}z_{\alpha\beta}z^{\alpha\beta})^{1/2}$. The resulting wavefunction is

$$\psi_{\mathrm{II}} = \exp(+\mathrm{i}\Delta\phi/\hbar)(\cos(\tfrac{1}{2}z) + \hat{z}_{\alpha\beta}G^{\alpha\beta}\sin(\tfrac{1}{2}z))\psi_{\mathrm{I}}(\mathrm{B}). \tag{4.4}$$

This is again a spin transformation related to a rotation through an angle z about the three-axis $z^\alpha = \frac{1}{2}\eta^{\alpha\sigma\beta\gamma}u_\rho z_{\beta\gamma}$.

Inserting into the Dirac current $j^\alpha(\mathrm{B})$ we obtain with $\psi(\mathrm{B}) = \psi_{\mathrm{I}}(\mathrm{B}) + \psi_{\mathrm{II}}(\mathrm{B})$

$$\begin{aligned} j^\mu(\mathrm{B}) = 2\{&[1 + (\cos(\tfrac{1}{2}z))(\cos(\Delta\phi/\hbar))]j_{\mathrm{I}}^\mu(\mathrm{B}) - (\sin(\tfrac{1}{2}z))(\sin(\Delta\phi/\hbar))\hat{z}^{*\mu}{}_\varepsilon S_{\mathrm{I}}^\varepsilon(\mathrm{B}) \\ &+ (\sin(\tfrac{1}{2}z))(\cos(\tfrac{1}{2}z) + \cos(\Delta\phi/\hbar))\hat{z}^\mu{}_\varepsilon j_{\mathrm{I}}^\varepsilon(\mathrm{B}) + (\sin(\tfrac{1}{2}z))^2\hat{z}^\mu{}_\rho\hat{z}^\rho{}_\varepsilon j_{\mathrm{I}}^\varepsilon(\mathrm{B})\}. \end{aligned} \tag{4.5}$$

The intensity of the two interfering waves behind the mixer is again given by $J(\mathrm{B})$ of (3.19). With (3.15*b*) and (3.18*b*) it follows from (4.5) as the exact result for the relative intensity in this particular global experiment:

$$\begin{aligned} J(\mathrm{B})/4J_{\mathrm{I}}(\mathrm{B}) = \tfrac{1}{2}\{&1 + (\cos(\tfrac{1}{2}z))(\cos(\Delta\phi/\hbar)) \\ &+ (1 + v_{\mathrm{I}}(\mathrm{B})^2)^{1/2}(\sin(\Delta\phi/\hbar))(\sin(\tfrac{1}{2}z))\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{S}}_{\mathrm{I}}(\mathrm{B})\}. \end{aligned} \tag{4.6}$$

(*b*) *Pure Lorentz boost.* In this case the global experiment results in $z_{\mu\nu} = 2\zeta_{[\mu}u_{\nu]}$ and the corresponding relations ($\hat{\zeta}^\alpha = \zeta^\alpha/\zeta$, $\zeta = (-\zeta^\alpha\zeta_\alpha)^{1/2}$)

$$(z_{\mu\nu}G^{\mu\nu})^{2m} = \zeta^{2m} \qquad (z_{\mu\nu}G^{\mu\nu})^{2m+1} = \zeta^{2m+1}2\hat{\zeta}_{[\mu}u_{\nu]}G^{\mu\nu} \tag{4.7}$$

($m = 0, 1, 2, \ldots$) lead with (4.2) to the wavefunction

$$\psi_{\mathrm{II}} = \exp(\mathrm{i}\Delta\phi/\hbar)(\cosh(\tfrac{1}{2}\zeta) + 2\hat{\zeta}_{[\mu}u_{\nu]}G^{\mu\nu}\sinh(\tfrac{1}{2}\zeta))\psi_{\mathrm{I}}(\mathrm{B}). \tag{4.8}$$

The Dirac current of the interfering waves in B follows as

$$\begin{aligned} j^\mu(\mathrm{B}) = 2\{&[1 + (\cosh(\tfrac{1}{2}\zeta))(\cos(\Delta\phi/\hbar))]j_{\mathrm{I}}^\mu(\mathrm{B}) + (\sinh(\tfrac{1}{2}\zeta))^2 u_\varepsilon j_{\mathrm{I}}^\varepsilon(\mathrm{B})u^\mu \\ &- (\sinh(\tfrac{1}{2}\zeta))(\sin(\Delta\phi/\hbar))\eta^\mu_{\alpha\beta\gamma}\hat{\zeta}^\alpha u^\beta S_{\mathrm{I}}^\gamma(\mathrm{B}) - (\sinh(\tfrac{1}{2}\zeta))^2\hat{\zeta}_\varepsilon j_{\mathrm{I}}^\varepsilon(\mathrm{B})\hat{\zeta}^\mu \\ &+ (\sinh(\tfrac{1}{2}\zeta))(\cosh(\tfrac{1}{2}\zeta) + \cos(\Delta\phi/\hbar))(\hat{\zeta}^\mu u_\varepsilon - \hat{\zeta}_\varepsilon u^\mu)j_{\mathrm{I}}^\varepsilon\}. \end{aligned} \tag{4.9}$$

For the measured relative intensity $J(\mathrm{B})$ we find with (3.18*a*)

$$\begin{aligned} \frac{J(\mathrm{B})}{4J_{\mathrm{I}}(\mathrm{B})} = \tfrac{1}{2}\{&(\cosh(\tfrac{1}{2}\zeta))(\cosh(\tfrac{1}{2}\zeta) + \cos(\Delta\phi/\hbar)) + (\sinh(\tfrac{1}{2}\zeta)) \\ &\times[\cosh(\tfrac{1}{2}\zeta) + \cos(\Delta\phi/\hbar)]\hat{\boldsymbol{\zeta}} \cdot \boldsymbol{v}_{\mathrm{I}}(\mathrm{B})\}. \end{aligned} \tag{4.10}$$

5. Applications

We will discuss the gravitational Aharonov–Bohm effect as a theoretical application of the results for the global case, and on the other hand interference experiments as an application of the formulae for the infinitesimal case.

Today's experimental possibilities are limited by the fact that the rotation ω^α can only be produced by the Earth's rotation and that the linear acceleration a^α can only be caused by Earth's gravity. The reason is that one cannot bring both interferometer and neutron source together (!) in a reference frame which accelerates or rotates relative to the Earth in an adjustable way. Because the Earth's rotation $\omega = 10^{-5}\,\mathrm{s}^{-1}$ and gravitational acceleration $a = 10^3\,\mathrm{cm\,s}^{-2}$ are extremely small, as far as our purposes are concerned, it will turn out that their influence on the interference pattern via the spinor amplitude can be completely neglected. Their influence via the phase shift, on the other hand, is measurable and has been detected by Collela *et al* (1975) for the gravitational acceleration and by Werner *et al* (1979) for the Earth's rotation. Nevertheless, to demonstrate how the results of § 3 have to be applied to experimental situations, we evaluate the intensity relations (3.20) including the terms which represent the influence of the spinor amplitude.

To do so, we assume an interferometer area of $\Sigma = 10\,\mathrm{cm}^2$ and neutrons with wavelength $\lambda = 1.4\,\text{Å} = 1.4\times10^{-8}\,\mathrm{cm}$, momentum $p = h/\lambda = 5\times10^{-19}\,\mathrm{g\,cm\,s}^{-1}$ and velocity $v = p/m = 3\times10^5\,\mathrm{cm\,s}^{-1}$.

5.1. Rotation

We assume vanishing curvature and torsion. The acceleration is caused by centrifugal forces only. The length of the paths of the two neutron waves in the interferometer may differ by Δl. In this case $z_{\kappa\lambda}$ of (3.7) reduces to $z_{\kappa\lambda} = (\omega_{\kappa\lambda} + 2a_{[\kappa}u_{\lambda]})$ which can be decomposed according to § 3.3 with $z_\mu = \omega_\mu \Delta s$, $\zeta = a_c \Delta s$, $\omega = (-\omega_\mu\omega^\mu)^{1/2}$ where a_c is the centrifugal acceleration. In $J(\mathrm{B})$ of (3.20) ζ introduces, via the centrifugal acceleration, a factor (rotational velocity/c), which is multiplied by $v_{\mathrm{I}}(\mathrm{B})/c$. We neglect this quadratic relativistic effect.

What remains to be determined is $\Delta\phi$ and Δs: (3.10) renders approximately $\Delta\phi/\hbar = 2m\omega\Sigma/\hbar + 2\pi\Delta l/\lambda = 0.31 + 4.5\times10^8\,\Delta l/\mathrm{cm}$ (with $\omega^\alpha \sim \Sigma^\alpha$).

This relation demonstrates the great influence of a difference Δl of path length on the phase. We have used the fact that with the normalisation (2.12) the centrifugal potential is approximately zero on the paths I and II. For the same reason the time delay (3.12) reduces to $\Delta s = -2\omega\Sigma/c^2 - \Delta l/v = -2.2\times10^{-25}\,\mathrm{s} - 3.3\times10^{-6}(\Delta l/\mathrm{cm})\,\mathrm{s}$ so that $\frac{1}{4}z = \frac{1}{4}\omega\Delta s = -5.5\times10^{-31} - 8.25\times10^{-12}\Delta l/\mathrm{cm}$. The contributions of the second and third term in (3.20) representing the influence of the Earth's rotation on the spinor amplitude are of this magnitude. Only the first term is measurable in (3.20) which is due to the phase difference $\Delta\phi$ alone.

5.2. Acceleration

We assume vanishing curvature, torsion and rotation which implies $\zeta = a\Delta s$, $z = 0$, $a = (-a_\mu a^\mu)^{1/2}$. In the discussion of the respective experiment, it has already been shown that the Earth's gravitational potential U causes a maximal contribution of $-8.5\times2\pi$ to $\Delta\phi/\hbar$ so that we derive from (3.10) $\Delta\phi/\hbar = -5.3 + 4.5\times10^8\,\Delta l/\mathrm{cm}$. For the time delay we may use (3.12): $\Delta s = -(7\times10^{-13} + 3\times10^{-6}\,\Delta l/\mathrm{cm})\,\mathrm{s}$. To obtain a maximal interference effect we assume that the interferometer is arranged so that $\cos(\Delta\phi/\hbar) = 1$ and $\hat{\boldsymbol{\zeta}}$ is parallel to $\boldsymbol{v}_{\mathrm{I}}$. The contribution of the influence of the Earth's gravitational acceleration on the spinor amplitude, as given by the second term in (3.20), is then of magnitude $\frac{1}{2}\zeta(\hat{\boldsymbol{\zeta}}\cdot\boldsymbol{v}_{\mathrm{I}}/c) = 5\times(10^{-26} - 10^{-19}\,\Delta l/\mathrm{cm})$. Again, the spin effect is of no practical importance and the pure phase effect remains.

5.3. *A gravitational Aharonov–Bohm effect*

In a metric theory of gravitation, a genuine gravitational field is related to a non-vanishing Riemann curvature tensor. The gravitational analogue of the well known electromagnetic Aharonov–Bohm effect is, therefore, the following. Although two coherent neutron waves move in regions where the curvature vanishes, there is, nevertheless, an effect on the resulting interference pattern from the non-vanishing curvature of the region from where they are excluded.†

Following Dowker (1967, 1969) we will discuss a cone with an apex. The two-dimensional cross-sections of a flat space–time described in cylindrical coordinates

$$ds^2 = dt^2 - d\rho^2 - \rho^2 d\varphi^2 - dz^2 \tag{5.1}$$

is given the topology of a cone in demanding

$$0 \leqslant \varphi \leqslant 2\pi/H \qquad H \leqslant 1 \tag{5.2a, b}$$

and identifying according to $(t, \rho, \varphi = 0, z) \leftrightarrow (t, \rho, \varphi = 2\pi/H, z)$. The limiting case $H = 1$ is the Minkowski space with Euclidean topology. The apex $\rho = 0$ does not belong to the space–time.

We use the notation $(x^0, x^1, x^2, x^3 = t, \rho, \varphi, z)$ and introduce the tetrad field

$$h^{\alpha}_{(0)} = \delta^{\alpha}_{0} \qquad h^{\alpha}_{(1)} = \delta^{\alpha}_{1} \qquad h^{\alpha}_{(2)} = \rho^{-1}\delta^{\alpha}_{2} \qquad h^{\alpha}_{(3)} = \delta^{\alpha}_{3} \tag{5.3a, b, c, d}$$

which is adjusted to the coordinate lines. The corresponding spinor connection is

$$\Gamma_{\alpha=0} = \Gamma_{\alpha=1} = \Gamma_{\alpha=3} = 0 \qquad \Gamma_{\alpha=2} = G^{(1)(2)}. \tag{5.4a, b}$$

The interferometer worldlines have tangent vectors $u^{\alpha} = h^{\alpha}_{(0)}$ which implies that there are no kinematical effects ($\omega_{\alpha\beta} = 0$, $a_{\alpha} = 0$). The two coherent neutron waves ψ_{I} and ψ_{II} pass around $\rho = 0$ (the resulting closed loop encloses the points $\rho = 0$.) We assume vanishing torsion and that the corresponding two paths are symmetric, so that there is no time delay and no phase difference ($\Delta s = 0$, $\Delta\phi = 0$), and the emission points A_0 and A_1 agree. With (5.4) we then obtain, as result of the integration in (2.19)

$$\psi'_{\mathrm{II}}(\mathrm{B}) = \exp(\Delta\varphi G^{(1)(2)})\psi_{\mathrm{I}}(\mathrm{B}) \qquad \Delta\varphi = 2\pi/H. \tag{5.5a, b}$$

That the spinor $\psi'_{\mathrm{II}}(\mathrm{B})$ of (5.5) does not already describe the second of the two interfering neutron waves in B can be seen directly in the limiting case $H = 1$ where the cone degenerates to a plane. In this case, with our assumptions above, we must have

$$\psi_{\mathrm{II}}(\mathrm{B}) = \psi_{\mathrm{I}}(\mathrm{B}) \tag{5.6}$$

which is not reflected by (5.5) because of $\Delta\varphi = 2\pi$. The reason for this is that the identical spin transformation is related to a Lorentz rotation of the tetrad through 0

† The gravitational AB effect has been considered for vectors as well as for spinors. Papini (1967) was the first to propose it for spinors: parallel transport of a spinor around a closed orbit which encloses a tube with non-zero curvature results in a non-zero spinor transformation. As for the vectorial case, the Regge calculus (Regge 1961) concerning cone-like manifolds is based on it. For spinors most of the calculations were done for the two-dimensional cone (embedded in a four-dimensional space–time) either with a smoothly capped dome giving a region of finite curvature (Ford and Vilenkin 1981), or with an apex giving a singularity (multiply connected manifold, Dowker 1967, 1969). To be complete, we mention that Lawrence *et al* (1973) have considered the massless Dirac equation in a weak external gravitational field to discuss the AB effect.

and 4π, and not through 2π. With regard to the spin structure, a tetrad which is obtained by a rotation through 2π must be regarded as a different object. (5.5) refers to such a tetrad. It needs the additional rotation by 2π or -2π to obtain the spinor $\psi_{\rm II}(\mathrm{B})$ which refers in this sense to the same tetrad in B as $\psi_{\rm I}(\mathrm{B})$:

$$\psi_{\rm II}(\mathrm{B}) = \exp(-2\pi G^{(1)(2)})\psi'_{\rm II}(\mathrm{B}). \tag{5.7}$$

Now (5.6) is fulfilled for $H = 1$.

The same must be observed in the general case $H > 1$. With (5.6) and (5.7) we then obtain

$$\psi_{\rm II}(\mathrm{B}) = \cos \pi\left(\frac{1-H}{H}\right) + 2G^{(1)(2)} \sin \pi\left(\frac{1-H}{H}\right) \tag{5.8}$$

and for the intensity with (4.6) finally

$$J(\mathrm{B}) = 2J_{\rm I}(\mathrm{B})\left[1 + \cos \pi\left(\frac{1-H}{H}\right)\right]. \tag{5.9}$$

The non-vanishing second term on the right-hand side represents a gravitational AB effect.

6. Conclusions

We have described the coherent neutron waves in the WKB approximation of the Dirac equation in Riemann–Cartan space. The interference result reflects influence on the spinor phase and on the spinor amplitude. The latter effect may not be neglected; torsion, for instance, acts only on the spinor amplitude.

The experimental interferometer arrangement has been discussed in detail. To obtain a stationary interference pattern, (i) the interferometer must move along a Killing congruence, (ii) the neutron source must be stationary, which results in Lie-propagated amplitudes and (iii) torsion must show the same space–time symmetry.

The influence of the interferometer kinematics and the external fields on the spinor amplitude can be read off most easily in the infinitesimal case. In general, there will be a time delay between the emission of the two coherent waves which finally interfere in one worldpoint. An influence of interferometer rotation and acceleration can be found only for non-vanishing time delay. It turns out that the left dual of the modified Riemann–Cartan curvature, the interferometer rotation and the three-axial vector of the torsion act like a spinorial Lorentz three-rotation. The interferometer acceleration (deviation from the gravitational free fall) and the modified curvature itself act like a spinorial Lorentz boost. A global evaluation of the exact expression can be used, for instance, to discuss the gravitational Aharonov–Bohm effect.

The interferometer itself does not react on space–time torsion. Neutron propagation, on the other hand, depends only on the total antisymmetric part of torsion. Accordingly, only this part influences the interference result either by a modified Riemann–Cartan curvature, or directly via the related three-axial vector of torsion which acts like an interferometer rotation. There is no influence of torsion on the phase difference or the time delay. Obviously, neutron interference offers, at least in principle, a possibility to measure space–time torsion.

Appendix 1. Time-like congruences

We repeat the definitions and theorems of the theory of time-like congruences with normalised tangent vector field $u^\alpha(x)$ (Synge 1960, Trautmann 1965)† on which the discussion of the stationary interferometer in § 2.2 is based.

Lemma 1 (*Ehlers* 1961). If a congruence $u^\alpha(x)$ is rigid, i.e.

$$\text{(i)}\ \theta=0 \qquad \sigma^{\alpha\beta}=0 \tag{A1.1}$$

the following conditions are equivalent:

$$\left.\begin{array}{l}\text{(ii)}\ (\omega^\beta{}_{;\varepsilon}u^\varepsilon)h^\alpha_\beta=0\\ \text{(iii)}\ (a^\beta{}_{;\varepsilon}u^\varepsilon)h^\alpha_\beta=\omega^\alpha{}_\beta a^\beta\end{array}\right\} \Leftrightarrow \text{(iv)}\ \partial_{[\alpha}a_{\beta]}=0. \tag{A1.2}$$

Condition (ii) means that the vector of the angular velocity is Fermi propagated, i.e. this vector does not rotate. The interpretation of (iii) is that the change in time of the acceleration vector a^α agrees with the propagation behaviour of the orthogonal connecting vector r^α (with $r^\varepsilon u_\varepsilon=0$) to the neighbouring worldline which, for a rigid congruence, is given by $(r^\beta{}_{;\varepsilon}u^\varepsilon)h^\alpha_\beta=\omega^\alpha{}_\beta r^\beta$. Condition (iii) means that the acceleration vector is permanently pointing to the same neighbouring particle.

If the worldlines of the congruence are trajectories of a one-parameter group of isometries of the space–time (Killing trajectories), the congruence is called isometric or stationary. The following lemma relates this with conditions (i) and (iv) of lemma 1:

Lemma 2 (*Salzmann and Taub* 1954)

$$\left.\begin{array}{ll}\text{(i)}\ \theta=0 & \sigma^{\alpha\beta}=0\\ \text{(iv)}\ \partial_{[\alpha}a_{\beta]}=0 & \end{array}\right\} \Leftrightarrow \begin{array}{l}\text{There exists a collinear vector field}\\ \xi^\alpha=\mathrm{e}^U u^\alpha \text{ with } \mathscr{L}_\xi g_{\alpha\beta}=0\end{array} \tag{A1.3}$$

and the acceleration has a potential

$$a_\alpha=-\partial_\alpha U \qquad U=\tfrac{1}{2}\ln(\xi^\varepsilon\xi_\varepsilon) \tag{A1.4a, b}$$

Lemma 2 shows that rigid congruences which in addition fulfil conditions (ii) and (iii) of (A1.2) are isometric and can only exist in a stationary space–time. An immediate consequence is $\mathscr{L}_\xi u^\alpha=0$ and

$$\partial_{[\beta}\xi_{\alpha]}=\mathrm{e}^U(\omega_{\alpha\beta}+2a_{[\alpha}u_{\beta]}). \tag{A1.5}$$

Appendix 2. Lie derivative of spinors

We have to give a brief introduction about the Lie displacement and the Lie derivative of spinors because these concepts are rarely used in the literature. Our treatment is

† Decomposition of $u_{\alpha;\beta}$ by means of the projection tensor $h_{\alpha\beta}=g_{\alpha\beta}-u_\alpha u_\beta$ leads to $u_{\alpha;\beta}=\omega_{\alpha\beta}+\sigma_{\alpha\beta}+\frac{1}{3}\theta h_{\alpha\beta}+a_\alpha u_\beta$ with $\omega_{(\alpha\beta)}=\sigma_{[\alpha\beta]}=\sigma^\alpha{}_\alpha=0$, $\omega_{\alpha\beta}u^\beta=\sigma_{\alpha\beta}u^\beta=0$. The quantities $\omega_{\alpha\beta}, \sigma_{\alpha\beta}, \theta$ defined this way describe respectively rotation, shear and expansion of a cloud of neighbouring particles with velocity u^α. The vector $a^\alpha=u^\alpha{}_{;\varepsilon}u^\varepsilon$, $a_\alpha u^\alpha=0$, $a=(-a_\alpha a^\alpha)^{1/2}$ is the acceleration (deviation from geodesic motion). $\omega^\alpha=\frac{1}{2}\eta^{\beta\alpha\kappa\lambda}\omega_{\kappa\lambda}u_\beta$, $\omega_\alpha u^\alpha=0$, $\omega=(-\omega_\alpha\omega^\alpha)^{1/2}$ represents the angular velocity of the rotation of neighbours with regard to Fermi-propagated axes in the rest space of a particle.

deliberately a heuristic one, focusing on the relation to the Lie displacement of a tetrad (for a more elaborate treatment see Kosmann (1972), compare as well Jhangiani (1977)). The corresponding intuitive understanding makes the mathematical description of a stationary working neutron source directly plausible.

In the following we treat various derivatives in a unified way. The respective definitions can all be based on displacements. Consider a field of tangent vectors $\boldsymbol{\xi}$ to curves parametrised by t, and let P be a point on a curve and P′ a point on the same curve at the infinitesimal 'distance' δt. In a local coordinate system this implies $x^{\alpha'} = x^\alpha + \xi^\alpha \delta t$.

Different specific displacements of a tetrad along the $\boldsymbol{\xi}$ curve can be introduced (parallel displacement, Lie displacement, Fermi displacement, . . .) and related here—with different vector derivatives $\mathscr{D}_\xi$ (directional derivative, Lie derivative, Fermi derivative, . . .). We assume that the metric $\boldsymbol{g}$ fulfils

$$\underset{\xi}{\mathscr{D}} \boldsymbol{g} = 0 \tag{A2.1}$$

so that a displaced tetrad remains a tetrad. This condition may be weakened but is sufficient for our purpose. To any such transport of a tetrad we can relate a displacement of a spinor, and accordingly define a respective spinor derivative $\mathscr{D}_\xi$ (spinor directional derivative, spinor Lie derivative, spinor Fermi derivative, . . .) in demanding that the components of the displaced spinor do not change during the transport if they are referred to the displaced tetrad. This process may be called co-dragging of a spinor. In the following we will use it to introduce the spinor Lie derivative.

Let $\boldsymbol{h}_a(\mathrm{P})$ and $\psi(\mathrm{P})$ be the tetrad and spinor in P and $\boldsymbol{h}'_a(\mathrm{P}')$ and $\psi'(\mathrm{P}')$ the transferred tetrad and the co-dragged spinor in P′ obtained by a certain displacement rule. The co-dragging is reflected by the numerical identity

$$\psi'(\mathrm{P}') = \psi(\mathrm{P}). \tag{A2.2}$$

$\psi'(\mathrm{P}')$ refers to the displaced tetrad $\boldsymbol{h}'_a(\mathrm{P}')$ and not to the tetrad $\boldsymbol{h}_a(\mathrm{P}')$ which will be found there. The respective derivations of the tetrad and spinor are then given by

$$\underset{\xi}{\mathscr{D}} \boldsymbol{h}_a = \delta t^{-1}[\boldsymbol{h}_a(\mathrm{P}') - \boldsymbol{h}'_a(\mathrm{P}')] \qquad \underset{\xi}{\mathscr{D}} \psi = \delta t^{-1}[\psi(\mathrm{P}') - \psi'(\mathrm{P}')^{\mathrm{L}}]. \tag{A2.3a, b}$$

The spinor $\psi'(\mathrm{P}')^{\mathrm{L}}$ is thereby obtained from $\psi'(\mathrm{P}')$ by the infinitesimal spin transformation which is related to the Lorentz transformation L.

$$\psi'(\mathrm{P}')^{\mathrm{L}} = (1 + \tfrac{1}{2} z_a{}^b G^a{}_b)\psi'(\mathrm{P}') \qquad \mathrm{L}\colon \boldsymbol{h}_a(\mathrm{P}') = (\delta^b_a + z_a{}^b)\boldsymbol{h}'_a(\mathrm{P}') \tag{A2.4a, b}$$

$(G^{ab} = \frac{1}{2}\gamma^{[a}\gamma^{b]})$ by which the displaced tetrad in P′ is rotated into the one that is already there. This spin transformation guarantees that the two spinors on the right-hand side of (A2.3b) refer to the same tetrad $\boldsymbol{h}_a(\mathrm{P}')$.

A direct consequence of (A2.3a) and (A2.4b) is

$$z_a{}^b = \delta t \langle \boldsymbol{h}^b, \underset{\xi}{\mathscr{D}} \boldsymbol{h}_a \rangle = h^b_\alpha (\underset{\xi}{\mathscr{D}} \boldsymbol{h}_a)^\alpha \delta t. \tag{A2.5}$$

Inserting (A2.4a) into (A2.3b) we obtain the final form of the spinor derivative.

$$\underset{\xi}{\mathscr{D}} \psi = \langle \boldsymbol{d}\psi, \boldsymbol{\xi} \rangle - \tfrac{1}{2}(\delta t)^{-1} z_a{}^b G^a{}_b \psi = (\partial_\alpha \psi)\xi^\alpha - \tfrac{1}{2} h^b_\alpha (\underset{\xi}{\mathscr{D}} \boldsymbol{h}_a)^\alpha G^a{}_b \psi \tag{A2.6a, b}$$

where we have used (A2.5) and made reference to a local coordinate system.

The covariant spinor derivative $\overset{\{\}}{\nabla}$ of the footnote to § 2.1 which refers to the Christoffel parallel propagation, can be read off immediately from (A2.6*b*) by specialising $\mathscr{D}_\xi$ to the directional derivative $\mathscr{D}_\xi = (\ )_{;\varepsilon}\xi^\varepsilon$. To obtain the Lie derivative of a spinor, we simply have to specialise on the right-hand side of (A2.6*b*) to the vector Lie derivative $\mathscr{D}_\xi = \mathscr{L}_\xi$ which leads to

$$\begin{aligned} \underset{\xi}{\mathscr{L}}\psi &= (\partial_\alpha\psi)\xi^\alpha - \tfrac{1}{2}h^b_{\ \beta}[(\partial_\alpha h^{\ \beta}_{a})\xi^\alpha - (\partial_\alpha\xi^\beta)h^{\ \alpha}_{a}]G^a{}_b\psi \\ &= (\partial_\alpha\psi)\xi^\alpha - \tfrac{1}{2}h^b_{\ \beta}[h^{\ \beta}_{a;\alpha}\xi^\alpha - \xi^\beta{}_{;\alpha}h^{\ \alpha}_{a}G^a{}_b\psi \\ &= \xi^\alpha\overset{\{\}}{\nabla}_\alpha\psi + \tfrac{1}{2}(\partial_{[\alpha}\xi_{\beta]})G^{\alpha\beta}\psi. \end{aligned} \tag{A2.7}$$

The Lie derivative of tensors is a pre-metric and pre-affine concept. Correspondingly, the explicit form of the Lie derivative of spinors also contains only partial derivatives. It may be convenient to rewrite it, for example, with the aid of the Christoffel connection as we have done above. The Lie derivative (A2.7) is type-preserving and fulfils all conditions of a derivative.

For the use in § 2 we note that according to the construction above, the Lie derivative vanishes ($\mathscr{L}_\xi\psi = 0$) if and only if the ψ field on the worldline with tangent vector ξ^α is obtained by co-dragging with a Lie displaced tetrad ($\mathscr{L}_\xi h^{\ \alpha}_{a} = 0$).

The Lie derivative of the adjoined spinor $\bar{\psi}$ follows from the demand that for scalars $\mathscr{L}_\xi$ reduce to the partial derivative

$$\underset{\xi}{\mathscr{L}}\bar{\psi} = \xi^\alpha\overset{\{\}}{\nabla}_\alpha\bar{\psi} - \tfrac{1}{2}\bar{\psi}(\partial_{[\alpha}\xi_{\beta]})G^{\alpha\beta} \qquad \underset{\xi}{\mathscr{L}}(\bar{\psi}) = \overline{\underset{\xi}{\mathscr{L}}\psi}. \tag{A2.8a, b}$$

The Dirac matrices γ^α are (1.0)-tensors and (1.1)-spinors. It can be shown that with (A2.1)

$$(\underset{\xi}{\mathscr{L}}\gamma)^\alpha = -g^{\alpha\kappa}(\underset{\xi}{\mathscr{L}}g_{\kappa\lambda})\gamma^\lambda = 0 \qquad \underset{\xi}{\mathscr{L}}\gamma^5 = 0. \tag{A2.9a, b}$$

A consequence of (A2.7)–(A2.9) is that for vectors and tensors like $j^\alpha = \bar{\psi}\gamma^\alpha\psi$ the spinor Lie derivative reduces to the one for tensors.

Appendix 3. Stationarity for the interference pattern

We prove a statement of § 2.2. It is to be expected that the interference pattern will not change in time, because of the Lie conditions (2.10), (2.13) and (2.14) which have to be fulfilled by apparatus, source and space–time, i.e.

$$\underset{\xi}{\mathscr{L}}\psi_{\mathrm{I}} \overset{\mathrm{M}}{=} 0 \qquad \underset{\xi}{\mathscr{L}}\psi_{\mathrm{II}} \overset{\mathrm{M}}{=} 0 \tag{A3.1a, b}$$

(where M denotes the worldline of the mixer) if one additionally requires the same symmetry for the space–time torsion:

$$\underset{\xi}{\mathscr{L}}K_{[\alpha\beta\gamma]} = 0. \tag{A3.2}$$

To show this it will not be necessary to assume that the expansion θ_v of the neutron beams vanishes.

The particle paths with tangent vector v^α are geodesic. Because of the commutativity of covariant and Lie derivative in a stationary space–time this leads to (Yano 1955)

$$\mathop{\mathscr{L}}_{\xi} v^\alpha = 0 \qquad \mathop{\mathscr{L}}_{\xi} \theta_v = 0. \tag{A3.3a, b}$$

Stationarity of the space–time metric also implies (Yano 1955)

$$\mathop{\mathscr{L}}_{\xi} \begin{Bmatrix} \gamma \\ \alpha\beta \end{Bmatrix} = 0 \qquad \mathop{\mathscr{L}}_{\xi} \Delta s_i = 0 \tag{A3.4a, b}$$

where Δs_i is the arc length of the neutron path between two arbitrary points lying on fixed lines of the u^α congruence.

Because of the propagation equations (2.3), (2.5) and (2.6) the spinors received in B are the following functions of the spinors emitted in A_0 and A_1:

$$\psi_{\mathrm{I,II}}(\mathrm{B}) = \mathscr{T} \exp\left(-\frac{im}{\hbar} \int_{\mathrm{A}_{0,1}}^{\mathrm{B}} \mathrm{d}s - \tfrac{1}{2} \int_{\mathrm{A}_{0,1}}^{\mathrm{B}} \theta_v \, \mathrm{d}s - \int_{\mathrm{A}_{0,1}}^{\mathrm{B}} \tilde{\Gamma}_\alpha v^\alpha \, \mathrm{d}s \right) \psi(\mathrm{A}_{0,1}). \tag{A3.5}$$

In choosing N particular u^α worldlines, we introduce an N-fold cutting of the paths I and II with related distances Δs_i, $i = 1 \ldots N$. Because of a property of the time-ordering operator, (A3.5) can then be written:

$$\begin{aligned} \psi_{\mathrm{I,II}}(\mathrm{B}) = \lim_{N\to\infty} \prod_{i=1}^{N} [1 - im\,\Delta s_i/\hbar - \tfrac{1}{2}\theta_v(s_i)\Delta s_i \\ - (\overset{\{\}}{\Gamma}_\alpha(s_i) + \tfrac{3}{2} K_{[\alpha\beta\gamma]}(s_i) G^{\beta\gamma}(s_i)) v^\alpha(s_i)\Delta s_i] \psi(A_{0,1}). \end{aligned} \tag{A3.6}$$

Finally using the product rule of the Lie derivative, the demand (A3.2), the results (A3.3)–(A3.4) and furthermore for $\psi(\mathrm{A}_0)$ and $\psi(\mathrm{A}_1)$ the fact that the neutron source is stationary, it follows that (A3.6) has the intended relations (A3.1) as an immediate consequence. The interference pattern is stationary.

Note added in proof. J Anandan (1979 *Nuovo Cimento* **53A**, 221) also considers an interference experiment based on an eikonal approximation of the Dirac equation. The influence on the spinor amplitude is represented by an additional phase difference which can only be evaluated in the low-energy limit. On the other hand, this influence results primarily in a Lorentz transformation. It is doubtful that the polarisation after the interference can be expressed as function of phase differences only. Furthermore, neither a theory of the interferometer nor a theory of the neutron source are developed. Accordingly, inertial effects due to the spin amplitude are not taken into account. The author states that torsion influences the phase via the 'closing gap'. This is wrong already for the following reason: the neutron paths are influenced by the Christoffel affinity only. Note that in this paper torsion is minimally coupled not to the Dirac Lagrangian but to the Dirac equation.

References

Anandan J 1977 *Phys. Rev.* D **15** 1448

Ashtekar A and Magnon A 1975 *J. Math. Phys.* **16** 342

Audretsch J 1981a *Grundlagenprobleme der modernen Physik* ed J Nitsch, J Pfarr and E-W Stachow (Bibliogr. Inst. Mannheim)

—— 1981b *Phys. Rev.* D **24** 1470

Bonse U and Hart M 1965 *Appl. Phys. Lett.* **6** 155

Bonse U and Rauch H 1979 *Neutron Interferometry* (Oxford: Clarendon)

Colella R, Overhauser A W and Werner S A 1975 *Phys. Rev. Lett.* **34** 1472

Dowker J S 1967 *Nuovo Cimento* B **52** 7669

—— 1969 *J. Phys. A: Math. Gen.* **2** 267

Eder E and Zeilinger A 1976 *Nuovo Cimento* B **34** 76

Ehlers J 1961 *Akad. Wiss. Lit. (Mainz) Abhandl. Math. Nat. Kl* **11** 793

Ford L H and Vilenkin A 1981 *J. Phys. A: Math. Gen.* **24** 2353

Greenberger D M and Overhauser A W 1979 *Rev. Mod. Phys.* **51** 43

Jhanghiani V 1977 *Found. Phys.* **8** 445

Klein A G, Opat G J, Cimmino A, Zeilinger A, Treimer W, Gähler R 1981 *Phys. Rev. Lett.* **46** 1551

Kosmann Y 1972 *Ann. Math. Pure Appl., Bologna* XCI 317

Lawrence J K, Leiter D and Szamosi G 1973 *Nuovo Cimento* B **17** 113

Papini G 1967 *Nuovo Cimento* B **52** 136

Rauch H and Petraschek D 1978 *Topics in Current Physics* No 6 (Berlin: Springer)

Rauch H, Treimer W and Bonse U 1974 *Phys. Lett.* **47A** 425

Rauch H, Wilfing A, Bauspiess W and Bonse U 1978 *Z. Phys.* B **29** 281

Rauch H, Zeilinger A, Badurek G, Wilfing A, Bauspiess W and Bonse U 1975 *Phys. Lett.* **54A** 425

Regge T 1961 *Nuovo Cimento* **19** 558

Salzmann G and Taub A H 1954 *Phys. Rev.* **95** 1659

Synge J L 1960 *Relativity: The General Theory* (Amsterdam: North-Holland)

Trautman A 1965 *Lectures on General Relativity* ed A Trautman, F A E Pirani, H Bondi (Englewood Cliffs, New Jersey: Prentice Hall)

Werner S A, Colella R, Overhauser A W and Eagan C F 1975 *Phys. Rev. Lett* **35** 1053

Werner S A, Staudenmann J-L and Colella R 1979 *Phys. Rev. Lett.* **42** 1103

Yano K 1955 *Theory of Lie derivatives* (Amsterdam: North-Holland)

Yasskin P B and Stoeger W R 1980 *Phys. Rev.* D **21** 2081

Zeilinger A 1976 *Z. Phys.* B **25** 97